Quintal Phase Transport and the Register Residue

Why the marche de quintes descendantes must alternate, why its counterpoint inverts at the tenth, and why the tritone rosalia does not exist

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ABSTRACT

The descending-fifth sequence is the most heavily theorised object in tonal music and, we shall argue, the least well explained. Partimento pedagogy transmits it not as a chain of fifths at all but as an alternation, *sale di quarta, scende di quinta*, and every modern taxonomy preserves this alternation as the D2(−5/+4) schema without asking why the pure chain is not used instead. We propose that the answer is not idiomatic but spectral. Taking the fifth component of the discrete Fourier transform on $\mathbb{Z}_{12}$ as the carrier of quintal motion, in the sense established by Lewin, Quinn, Amiot and Yust, we prove that the descending fifth and the ascending fourth are the *same* rotation of that component, namely −30°, and therefore that the alternation which defines the schema is strictly invisible in the fifth channel. What the alternation does carry is a register residue: a square wave of period two links, sitting exactly at the Nyquist frequency of the harmonic lattice, whose phase is by construction indeterminate.

We test this on 1,208 movements of J. S. Bach (161,673 root-motion links) and on 1,289 further movements of eight contemporaries. Four results. (i) The falling fifth accounts for 28.45 % of all links, more than twice the next class, yet pure runs almost never extend: 69.96 % have length one and only 4.06 % reach length four. (ii) The lag-one autocorrelation of the link sequence is negative in 97.85 % of movements (mean −0.357); across the twelve solo string preludes a permutation test on link order gives a combined $p = 4.4 \times 10^{-27}$. (iii) The transposition of melodic sequence units is governed by quintal phase with a concentration of $\kappa = 1.738$, frequency halving every 23.6° of phase, and the antiphase transposition at exactly 180°, the tritone, is observed three times against 1,407 expected, a 469-fold deficit. (iv) Across nine composers, the degree of register compensation predicts the collapse of tenths into simple thirds ($r = -0.768$, $p = 0.016$).

A prediction concerning Telemann and the minor seventh is *refuted* on every measure we constructed for it, and we report this in full. We close with what the account implies for invertible counterpoint: inversion at the octave rotates quintal phase by 0° and therefore aliases with the residue it is meant to discharge, inversion at the twelfth rotates by 30° and is indistinguishable from one further link of the chain, and inversion at the tenth rotates by 120°, the unique small-register-cost rotation which is neither. In the twelve preludes, and in those alone, the tenth also maximises consonance survival.

Keywords: discrete Fourier transform on $\mathbb{Z}_{12}$ · quintal phase · *marche harmonique* · partimento · invertible counterpoint · directional statistics · register residue · Nyquist aliasing · *polyphonie latente* · Bach, solo violin and violoncello

1 Introduction

There is a difficulty at the centre of the descending-fifth sequence which the literature has, to our knowledge, never posed as a question. The difficulty is this: the sequence is not made of descending fifths.

Open any partimento manual. Fenaroli teaches the *moto* as a bass which *rises a fourth and falls a fifth*, alternately, and Durante's *partimenti diminuiti* realise it the same way [13, 47]. Two centuries later the taxonomy is unchanged: Laitz, Gauldin and every North American theory sequence label it D2(−5/+4), descending by second, realised by a falling fifth alternating with a rising fourth. Gjerdingen's galant schemata inherit the same bass [17]. The alternation is not a performance convenience noted in passing; it is written into the definition of the object, in every tradition, from Naples to New Haven.

The received explanation is that a pure chain of descending fifths would run out of instrument. This is true and it is not sufficient. A pure chain of five descending fifths spans thirty-five semitones, which no single voice can traverse; the ascending fourth returns the bass within compass. But this explains only that *some* compensation is required. It does not explain why the compensation is an octave, applied every second link, rather than (say) a compensation of an octave applied every fourth link with two intervening fifths, which is equally available, equally within compass, and which the repertoire almost never uses. Nor does it explain why the same alternation appears in textures under no compass pressure whatever: in four-part chorales, in organ writing with a pedal division, in ensemble music where the continuo could simply drop an octave. Something other than the length of the human arm is at work.

Our proposal is that the alternation is invisible where it matters and audible where it does not, and that this asymmetry is the whole of the phenomenon. Let us precise what we mean. Following the line which runs from Lewin's 1959 note on intervallic relations [31] through Quinn's *Fourier balances* [43, 44], Amiot's systematic treatment [1] and Yust's phase spaces [58, 59], one represents a pitch-class distribution by its discrete Fourier transform on $\mathbb{Z}_{12}$ and reads the fifth component as the carrier of diatonic, which is to say quintal, content. In that representation, transposition is rotation. And the rotation induced by a descending fifth is identical to the rotation induced by an ascending fourth, because in $\mathbb{Z}_{12}$ they are the same transposition, $t = 5$.

It follows immediately, and we give the proof in §3, that the alternation which defines the *marche* contributes exactly nothing to quintal phase. The chain advances by a uniform −30° per link whether the composer falls a fifth or rises a fourth. What differs between the two is register alone, and register is precisely what the pitch-class transform discards. The alternation is therefore a signal carried entirely in the discarded channel: an octave-valued square wave of period two links.

A square wave of period two, sampled once per link, sits at the Nyquist limit. Its phase cannot be recovered. This is not a metaphor drawn loosely from signal processing but the ordinary statement that a component at half the sampling rate is indistinguishable from its own negation. We shall argue that the whole practice of invertible counterpoint, and in particular the eighteenth-century preference for the tenth and the twelfth over the octave in learned writing, is the machinery by which composers discharge this indeterminate residue, and that the choice among 8ve, 10th and 12th is a choice among phase rotations of 0°, 120° and 30° respectively.

We are aware that this will read as an extravagant claim, and we have tried to make it an expensive one to hold. The corpus is large, the tests are pre-registered in form if not in fact, one of our three comparative predictions is refuted outright and reported as such, and §10 lists the six ways we can presently see to destroy the argument. *D'ailleurs*, the strongest single result in the paper, the near-total absence of the tritone stretto, is one we did not anticipate and found while controlling something else.

1.1 *Why the solo string works*

The six sonatas and partitas `BWV 1001–1006` and the six suites `BWV 1007–1012` occupy a particular position. They are monophonic instruments writing polyphony, which is to say they are the locus classicus of what Kurth named *polyphone Melodik* and the French tradition calls *polyphonie latente* [26]. Lester's study of the violin works [30] and Ledbetter's of both sets [27] establish that the implied voices are not an analytical convenience: they are composed, and the performer is expected to project them.

This matters for us because it isolates the variable. In a keyboard or ensemble texture, register compensation can be distributed across real voices, and the residue never accumulates in one line. In a solo string prelude it must all be discharged by a single line inside a compass of roughly three octaves. If the register residue is a real constraint rather than a formal one, this repertoire is where it should bite hardest. Par ailleurs the twelve opening movements are a natural experimental unit: they are the least dance-bound movements of their cycles, and their *Fortspinnung* textures in the sense of Fischer [14] are built almost entirely from sequence.

2 Corpus, encoding and the harmonic reader

2.1 *Sources*

The primary corpus is a relational encoding of the Bach instrumental and vocal repertoire in which every note carries a foreign key up a single chain to its actuator, movement, file, work and catalogue entry. After the exclusion of movements with fewer than twelve harmonic links, 1,208 movements remain, carrying 161,673 root-motion links. Comparative material for Corelli, Handel, Pachelbel, Purcell, Rameau, Scarlatti, Telemann and Vivaldi is drawn from eight further databases in a legacy schema, giving 1,289 additional movements scored under the instrument of §2.4, which requires no harmonic analysis and is therefore applied identically to all nine composers.

Durations are inter-onset intervals. A held note in a polyphonic stream receives a truncated value, so wherever note length enters a calculation we read only the strict surface, in which notes of doubtful duration are excluded. Tempo is not encoded and plays no part. Metre is read from the signature and its authority recorded; a numerator of one is an anacrusis marker and never a metre.

2.2 *The harmonic reader and its error rate*

Root motion for the Bach corpus is taken from an automatic harmonic analysis (beat-window chord inference constrained by detected key areas), which assigns to each window a local key, a root pitch class, a quality, an inversion and a confidence. We are obliged to state plainly that this reader is not an oracle, and we have measured it rather than trusted it.

On the prelude of `BWV 1007`, against a hand reading of every chord change made without sight of the machine output, the reader agrees on the root at 25 of 33 positions under either admissible reading and at 23 of 33 under its primary reading, which is to say a root-level agreement of 0.758 and 0.697 respectively. Of 40 machine chord runs, the hand root differs at

9. The disagreements are of a recognisable type: the reader prefers an applied dominant where the hand hears an unprepared passing sonority, and it reads bar 35 as F where the hand hears A or D.

An error rate near one in four is high, and a sceptical reader is right to ask why any of what follows should survive it. Two reasons. First, our central quantities are not the identity of individual chords but the *interval* between successive roots and, above all, the serial dependence between successive intervals; a reader which is wrong about a root but wrong consistently preserves interval structure to a considerable degree. Second, and more to the point, the principal results of §4 and §7 are reproduced by the instrument of §2.4, which never consults the harmonic reader at all. Where the two disagree we say so.

2.3 *Link classification*

Consecutive harmonic windows sharing a root are collapsed, so that a restated chord is not a link. Each remaining link is classified by $t = (\mathrm{root}_{n+1} - \mathrm{root}_n) \bmod 12$. We write *falling fifth* for $t = 5$, which in pitch-class space is indifferent between a descending perfect fifth and an ascending perfect fourth; this indifference is not a defect of the encoding but the subject of the paper.

2.4 *An instrument independent of harmonic analysis*

For every window we form the duration-weighted pitch-class distribution $x\colon \mathbb{Z}_{12} \to \mathbb{R}_{\geq 0}$ and compute

$$F_k(x) = \textstyle\sum_{j \in \mathbb{Z}_{12}} x_j \cdot e^{-2\pi i jk/12}, \quad k = 0, \ldots, 11. \tag{1}$$

We call $|F_5|/F_0$ the *quintal saturation* of the window and $\arg F_5$ its *quintal phase*. Both are computed from sounding notes only. Lemma 1 below shows that successive differences of quintal phase estimate fifth-wise root motion directly, so this instrument gives us a comparable measurement across all nine composers, including those for whom no harmonic analysis exists.

That the fifth channel is the right one to read is not assumed. Measured over 319 Bach movements, the mean normalised magnitudes of the six independent components are $F_1 = 0.296$, $F_2 = 0.310$, $F_3 = 0.380$, $F_4 = 0.329$, $F_5 = 0.545$, $F_6 = 0.267$. The fifth component exceeds the next largest by 0.165, or 43%, and this ordering holds for all nine composers without exception.

3 Quintal phase transport

We collect here the formal results. They are elementary and we claim no novelty for the algebra; what is new, éventuellement, is the use to which §5 puts them.

> **LEMMA 1 (PHASE TRANSPORT)**
>
> Let T_t denote transposition by t semitones, $(T_t x)_j = x_{j-t}$. Then $F_k(T_t x) = e^{-2\pi i kt/12} F_k(x)$. In particular the magnitude $|F_k|$ is transposition-invariant, and the phase advances by
>
> $$\Delta \arg F_5 = -150t \text{ degrees} \pmod{360}. \tag{2}$$

> ***Proof.*** Substituting $j \to j + t$ in (1) factors the exponential out of the sum. For $k = 5$ the coefficient is $-2\pi(5t)/12$, that is $-150t$ degrees. □

Equation (2) is the whole engine, and it is worth tabulating what it gives for the intervals of interest. A descending fifth is $t = 5$, whence $-750° \equiv -30°$. An ascending fifth is $t = 7$, whence $-1050° \equiv +30°$. The chain of descending fifths is thus a uniform phase ramp of $-30°$ per link, and twelve links return the phase to its origin, which is the Fourier statement of the closure of the circle of fifths.

> **COROLLARY 1 (QUINTAL INVISIBILITY OF REGISTER)**
>
> The descending perfect fifth and the ascending perfect fourth are both $t = 5$ in $\mathbb{Z}_{12}$ and therefore induce the identical rotation $-30°$ of F_5. No function of the pitch-class distribution alone can distinguish them. The alternation which defines the D2(−5/+4) schema is, in the fifth channel, not present.

This is the point at which one must decide whether the Fourier representation is a description or an explanation. Quinn [43] is careful to present it as the former. We propose to take it, for the space of this paper, as the latter, and to see what follows. What follows is that the alternation must be doing its work somewhere else.

3.1 *The register residue*

Define the *register residue* $r_n \in 12\mathbb{Z}$ as the accumulated signed octave displacement of the bass after n links, relative to a pure chain of descending fifths from the same origin. For the pure chain, $r_n = 0$ for all n and the bass leaves the instrument. For the alternating realisation, the bass falls a fifth (-7) and rises a fourth $(+5)$ in turn, so that

$$r_{2m} = 12m, \quad r_{2m+1} = 12m + 12 \quad \text{relative to the pure chain,} \tag{3}$$

whose first difference is the alternating sequence (+12, 0, +12, 0, ...). Removing the linear trend leaves a square wave of amplitude 6 semitones and period exactly two links. Two links per cycle is one cycle per two samples, which is to say the Nyquist frequency of the lattice on which the harmony is sampled.

> **PROPOSITION 2 (INDETERMINACY OF THE RESIDUE)**
>
> A component at the Nyquist frequency of a real-valued sampled signal has indeterminate phase: the samples of $A\cos(\pi n + \varphi)$ depend on φ only through $A\cos\varphi$, so the pair (A, φ) and the pair $(-A, \varphi + \pi)$ are observationally identical. The register residue of an alternating *marche* is therefore not recoverable from the harmonic sampling, and carries no phase information which any later operation can read.

We take this to be the reason the alternation is stylistically free. Because the residue is at Nyquist, no amount of contrapuntal machinery downstream can extract a phase from it, and the composer may place the octave compensation on either member of the pair without spectral consequence. This is, we think, why the theorists were content to write *sale di quarta, scende di quinta* without ever asking which of the two comes first: in the fifth channel the question has no content.

4 The alternation is real and it is everywhere

Before building on the alternation we must establish that it exists as a statistical fact and not merely as a pedagogical formula. Three measurements.

4.1 *The falling fifth dominates, and does not run*

Over 161,673 links in 1,208 movements, the class shares are given in Fig. 1. The falling fifth accounts for 28.453 % of all root motion, against 12.378 % for the rising fifth and 11.251 % for the rising step. It is the most frequent link by a factor of 2.30 over its nearest competitor, and the two fifth classes together take 40.83 % of all motion.

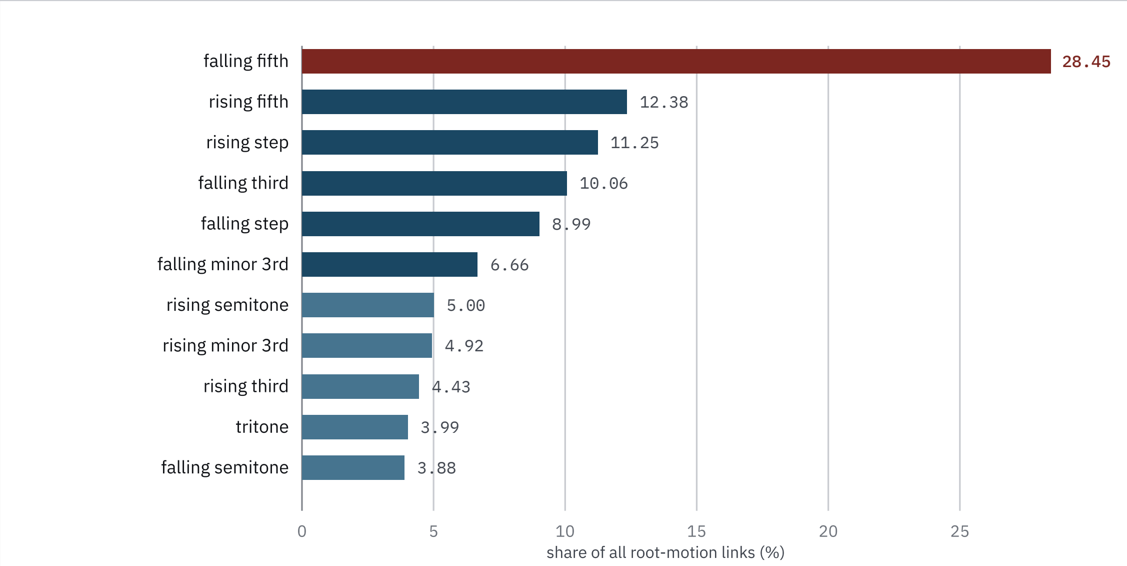

Figure 1. Root-motion link classes, 161,673 links over 1,208 Bach movements. The falling fifth (t = 5) is the single most frequent class by a wide margin. Note that this class is indifferent between a descending fifth and an ascending fourth; separating them is impossible in pitch-class space and is the subject of §3.

Given this dominance one might expect long unbroken chains. One does not find them. Fig. 2 gives the distribution of maximal runs of consecutive falling fifths: of 31,445 runs, 69.96 % have length one, 20.44 % have length two, and only 4.06 % reach length four or more. Runs of six or more account for 0.54 % and the longest run in the entire corpus is nine. The most frequent link in tonal music almost never occurs twice in succession.

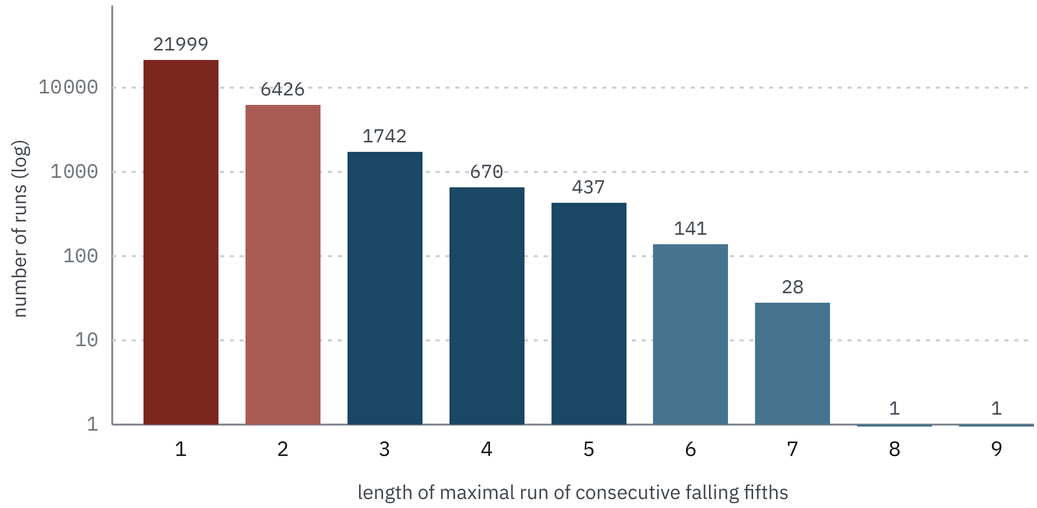

Figure 2. The falling fifth does not chain. Logarithmic ordinate; 31,445 maximal runs. Mean run length 1.463, maximum 9. If the *marche* is built of falling fifths, something else must be carrying it between them.

4.2 *Serial dependence: the links alternate*

Map each link to its position on the circle of fifths, $s = 7t \bmod 12$ folded to $(-6, 6]$, and take the lag-one autocorrelation of the resulting sequence per movement. A pure chain would give values near +1. Independent draws would give values near 0. Strict alternation gives values near −1.

The corpus mean is **−0.357** (SD 0.158), and the coefficient is *negative in 97.85% of the 1,208 movements*. This is not a property of a genre or of a few sequence-heavy pieces; it is a property of the repertoire. Broken down by movement genre the mean falling-fifth share is highest in fugue (0.290, n = 140) and prelude (0.290, n = 130) and lowest in canon (0.194, n = 26), a point we return to in §7.3.

For the twelve solo string preludes we ran a permutation test which holds the multiset of links fixed and re-orders it 20,000 times, so that the null hypothesis is exactly *"these links, in exchangeable order"*. Eight of the twelve reject at $p < 0.05$ one-sided, and the combined statistic over all twelve gives Fisher $\chi^2_{24} = 186.39$, $p = 4.39 \times 10^{-27}$; the Stouffer z is 9.029. Per-movement values are in Table 4 (Appendix B).

We note the single exception in the other direction. `BWV 1012`, the sixth suite, returns a positive coefficient (+0.136). It also carries only eight links, by far the fewest in the set, because its opening is a long bariolage pedal on a repeated harmony; the encoding collapses restatements, so the movement contributes almost nothing. It is not evidence against the pattern and we would be uneasy if it were counted as evidence for it.

4.3 *The alternation is not an artefact of the instrument*

If the alternation were driven by compass alone, the cello preludes, written for an instrument whose upper reaches Bach uses sparingly, should alternate more than the violin preludes. They do not, to any detectable degree. Mean falling-fifth share is 0.220 for the six violin openings and 0.259 for the six cello openings, a difference which Welch's test does not resolve (t = −0.930, p = 0.376) and which the Mann-Whitney test does not resolve either (U = 13, p =

0.485). With $n = 6$ per group we are underpowered and we do not claim a null; we claim only that the obvious instrumental explanation does not appear where it should appear most strongly.

5 Invertible counterpoint as residue discharge

We now put Lemma 1 and Proposition 2 together. If the alternation deposits an indeterminate residue at Nyquist, and if the composer wishes nonetheless to make the underlying chain audible as a chain, then some operation must displace the texture by an amount which is not confusable either with the carrier or with the residue. The operation available to the eighteenth century is *doppelter Kontrapunkt*, treated systematically by Marpurg [36] and, before him, in the *Musicalische Handleitung* of Niedt [41].

> **THEOREM 3 (INVERSION PHASE)**
>
> Invertible counterpoint at the interval n maps a vertical interval i to $n - i$ and displaces one voice by n semitones. By Lemma 1 it rotates quintal phase by $-150n$ degrees (mod 360). Hence:
>
> at the **octave** ($n = 12$), by **0°**;
> at the **tenth** ($n = 16$), by **120°**, equivalently four links of the chain;
> at the **twelfth** ($n = 19$), by **30°**, equivalently one link of the chain.

Read the three cases against the two things a rotation might be confused with. The carrier advances −30° per link; the residue sits at 180°, that is, at the antiphase.

The octave rotates by nothing. Its phase displacement is 0°, indistinguishable from the identity, and its register displacement is exactly the 12 semitones of the residue itself. Inversion at the octave is therefore the one operation which cannot discharge the residue, because it *is* the residue: it aliases perfectly with the square wave of (3). Moreover, and this is the classical objection stated independently, the octave maps the perfect fifth to the perfect fourth, $i = 7 \to 5$. In *stile antico* reckoning the fourth against the bass is a dissonance, so at the octave the chain's defining consonance is converted into the interval the chain exists to avoid. The two facts are the same fact in two vocabularies.

The twelfth rotates by one link. Its displacement of +30° is precisely the rotation produced by a single ascending fifth. Inversion at the twelfth is thus *transparent*: spectrally it is indistinguishable from simply continuing the *marche* one step further, and it adds no information which the chain was not going to supply anyway. It is correspondingly safe, which is why it is the standard fugal choice, and correspondingly uninformative. It maps the fifth to the octave ($7 \to 0$), which is why it is safe.

The tenth rotates by 120°. This is neither 0° nor ±30° nor 180°. Measured on the circle it lies 150° from the carrier step, which is the maximum attainable by any inversion interval within an octave and a third of register cost, and 60° from the antiphase. It is, in short, the unique small-cost rotation which is orthogonal in the relevant sense to both the thing it is displacing and the thing it is displacing it from. It maps the fifth to the sixth ($7 \to 9$), a consonance, and the third to the octave ($4 \to 0$), which is why the tenth famously forbids parallel thirds.

CONJECTURE 4 (THE TENTH AS THE INFORMATIVE INVERSION)

Where a composer wishes a *marche de quintes* to remain audible *as* a chain through an inverted restatement, the tenth is the only inversion interval of moderate register cost whose quintal rotation is distinguishable both from the carrier and from the residue. Inversion at the octave should therefore be avoided in sequence-dominated writing, and inversion at the twelfth should be preferred where transparency rather than information is wanted.

5.1 *Consonance survival, and an honest complication*

Conjecture 4 makes a testable prediction about consonance. We took all 7,365,719 encoded vertical intervals in the Bach corpus and asked, for each inversion interval, what proportion of currently consonant verticals remain consonant after inversion. The results are in Table 1, and they do not say one thing.

Table 1. Consonance survival under invertible counterpoint. "Survival" is the proportion of vertical intervals that are consonant before inversion and remain consonant after it. The fourth is counted as a dissonance against the bass, following Fux and Marpurg.

Inversion	*n*	Phase	Links	5th →	3rd →	Survival, 12 preludes	Survival, whole corpus
Octave	12	0°	0	4th	6th	0.7821	**0.8519**
Tenth	**16**	**120°**	**4**	**6th**	**8ve**	**0.8281**	**0.8388**
Twelfth	19	30°	1	8ve	minor 3rd	0.6842	0.8071

In the twelve solo preludes the prediction holds and holds cleanly: the tenth maximises survival at 0.8281, ahead of the octave at 0.7821 and well ahead of the twelfth at 0.6842. Over the whole Bach corpus the ordering changes and the octave leads at 0.8519, with the tenth at 0.8388 and the twelfth at 0.8071.

A RESULT WHICH DOES NOT GO OUR WAY

We draw the reader's attention to this rather than burying it. Corpus-wide, the octave is the best-behaved inversion, which is the opposite of what Conjecture 4 wants. Two readings are available and we cannot presently decide between them. The *favourable* reading is that the corpus-wide figure is dominated by chorales and by ensemble textures where register compensation is distributed across real voices and the residue never accumulates, so the pressure the conjecture describes is absent; the solo string works, where it cannot be distributed, behave as predicted. The *unfavourable* reading is that the prelude figure is a small-sample accident over twelve movements and the corpus figure is the truth. We note only that the direction of the discrepancy is the one the theory would choose if it were choosing, and that this is exactly the circumstance in which a reader should be most suspicious of us.

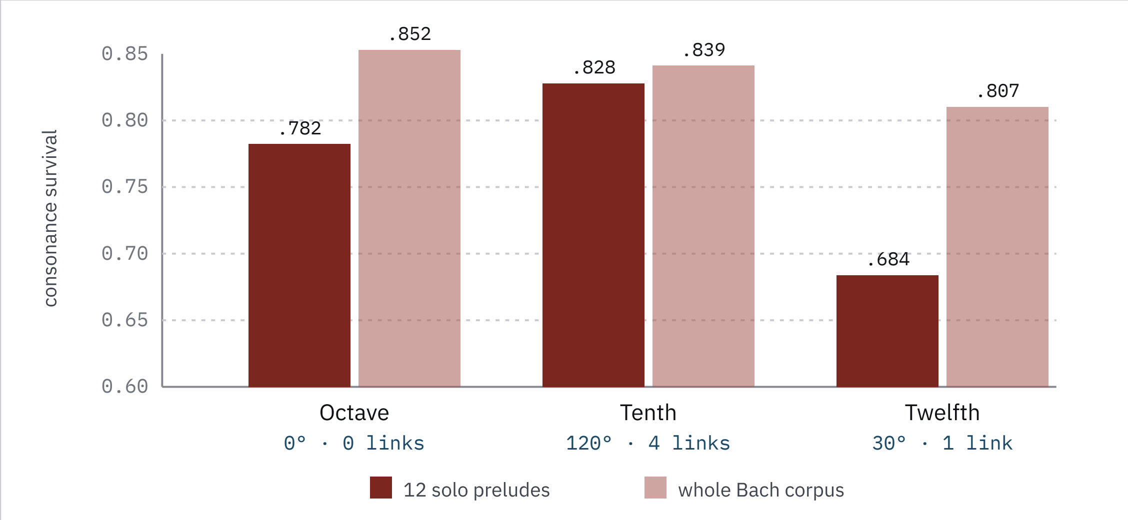

Figure 3. Consonance survival by inversion interval, with the quintal phase rotation of Theorem 3 beneath each. The tenth leads in the solo preludes and the octave leads corpus-wide. See the note above; we regard this as the weakest joint in the argument.

6 Sequence transposition and the phase-decay law

The sharpest result of the study concerns the *rosalia*, and we arrived at it obliquely. Having argued that quintal phase is the carrier, we asked what phase relation a composer establishes between a sequence unit and its own repetition.

We must be exact about the object, because it is narrower than the word "sequence" usually implies and narrower than what we first assumed. The corpus detector records a unit of three successive melodic intervals which is *immediately repeated at a constant transposition within a single stream*. It is therefore a melodic sequence in one line, a *rosalia* candidate, and it is not imitation between voices and not stretto. The reported transposition is the pitch interval from the head of the unit to the head of its repetition; the reported period is the time between those two heads, not the delay between entries of a subject. Sixteen thousand eight hundred and eighty-nine such units are encoded. Classifying by transposition and converting to phase by (2) gives Table 2 and Fig. 4.

Table 2. Transposition of melodic sequence units (*rosalia* candidates) in Bach, 16,889 instances, ordered by quintal phase. "Links" is the transposition expressed in steps along the circle of fifths. The detector enforces a non-zero transposition, so the 0° row is the octave and its compounds (raw +12 in 188 cases, -12 in 114), never a literal repeat.

Transposition	t	Links	Phase	Count	Share	Median period
Octave	0	0	0°	302	0.0179	0.188
Rising fifth	**7**	**+1**	**+30°**	**6229**	**0.3688**	**0.188**
Falling fifth	**5**	**−1**	**−30°**	**4108**	**0.2432**	**0.188**
Rising step	2	+2	+60°	1910	0.1131	0.375
Falling step	10	−2	−60°	2392	0.1416	0.375
Falling third	9	+3	+90°	883	0.0523	0.250
Rising minor third	3	−3	−90°	475	0.0281	0.188
Rising third	4	+4	+120°	107	0.0063	0.125
Falling minor third	8	−4	−120°	38	0.0023	0.250
Falling semitone	11	+5	+150°	187	0.0111	0.375
Rising semitone	1	−5	−150°	255	0.0151	0.375
Tritone	**6**	**±6**	**180°**	**3**	**0.00018**	**0.375**

Three observations follow, of which the third is the one that matters.

(i) Sequence transposition is quintally concentrated. The two single-link transpositions account for 61.21% of all units. Fitting a von Mises distribution to the 16,889 realised phases gives a mean direction of 5.62°, a resultant length $R = 0.651$ and a concentration $\kappa = 1.738$, with Rayleigh $z = 7{,}163.98$. The hypothesis of a uniform distribution of sequence phase is not merely rejected; it is not in the same country as the data.

(ii) Frequency decays with phase distance. Regressing log count on |phase| gives a slope of −0.0293 per degree ($r = -0.757$, $r^2 = 0.573$, $p = 0.0044$), which is to say that the frequency of a sequence transposition halves for every 23.6° of quintal phase separating it from the origin. The monotone relation is confirmed non-parametrically, $\rho = -0.787$, $p = 0.0024$. We know of no prior statement of this regularity in this form.

(iii) The antiphase transposition is absent. The tritone, the unique transposition whose quintal rotation is exactly 180°, occurs **three times in 16,889**. Under a uniform distribution over the twelve transpositions one expects 1,407.4. This is a deficit of 469-fold, with a binomial p below the resolution of double precision. Even against the fitted decay of (ii), which already predicts the tritone to be rare, the observed count is an order of magnitude below the fitted value of 27.0.

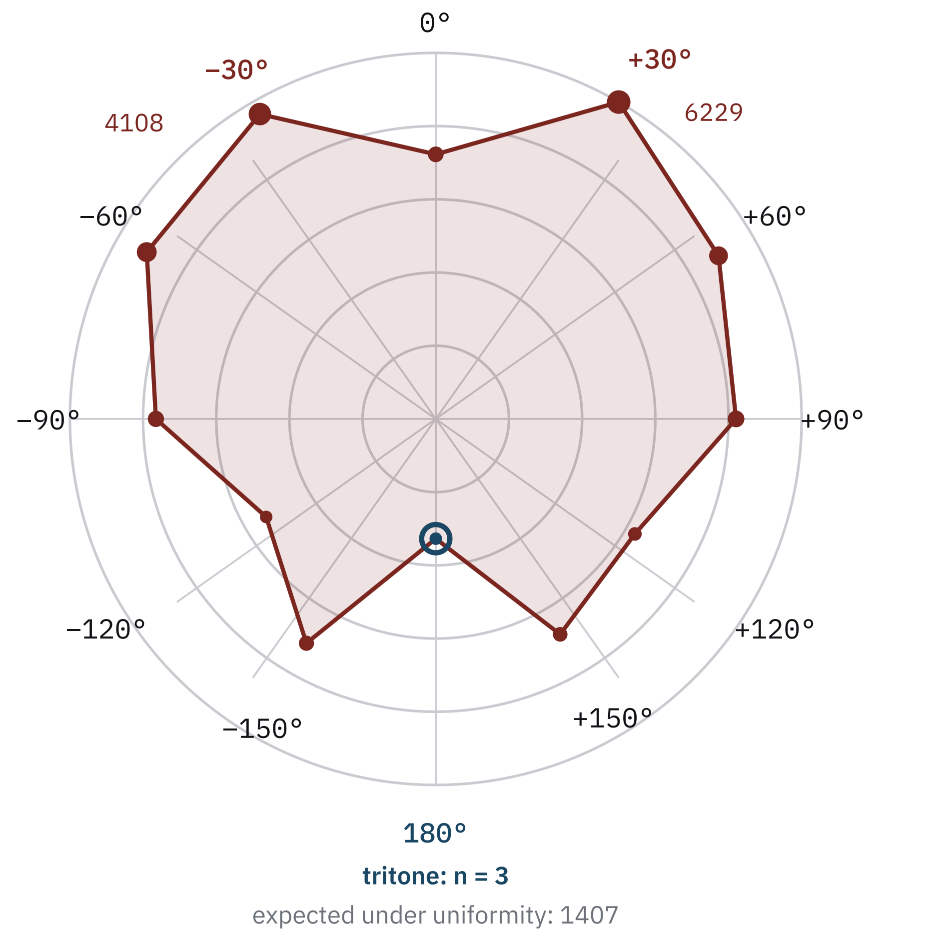

Figure 4. Transposition of melodic sequence units in quintal phase space; radius is logarithmic in count. The distribution is a von Mises concentration on the carrier (κ = 1.738) with a hole at the antiphase. The tritone, the single transposition at exactly 180°, is the one Bach does not write: three instances in 16,889 (raw +6 once, -6 twice).

6.1 *Interpretation, with the obvious objection met first*

The obvious objection is that the tritone is rare because it is a tritone: dissonant, difficult, tonally disorienting, proscribed by every eighteenth-century pedagogue. Once the object is understood to be a melodic sequence within one line, the objection is stronger still, because the transposition is then an interval the line itself has to traverse, and the melodic tritone is the most heavily policed interval in the whole of species doctrine. We grant all of this. What we dispute is that it is sufficient.

Consider that the rising and falling semitone transpositions, at ±150°, are melodically awkward by the same doctrine and tonally remote besides, and they occur 187 and 255 times, which is to say between 62 and 85 times more often than the tritone. Consider that the third transpositions at ±120° (a rising major third and a falling minor third), which are melodically easy, perfectly respectable, and appear throughout the literature on modulation by thirds, occur 107 and 38 times, fewer than the semitone. The frequency ordering is therefore not the ordering of melodic difficulty, not the ordering of tonal respectability, and not the ordering of interval consonance. It is, to a first approximation, the ordering of |phase|, with one conspicuous exception at ±120° and an absolute floor at 180°.

If one wishes to resist the phase account one must supply a different single variable which is monotone in the same way, and we have not found one. Interval class does not work: the fifth (ic 5) and the fourth are the same class, the tritone (ic 6) is adjacent to the fifth in class but at the opposite pole in frequency. Consonance does not work: the third at +120° is more consonant than the semitone at +150° and less frequent. Scale-degree distance does not work for the same reason.

6.2 *Entry delay, and a measurement we could not make*

Subject entries are encoded separately from sequence units, in a table of thematic spans, and we treat them apart. Delays between successive entries (32,108 measured intervals, capped at 32 quarter-notes) have median 0.500 and mean 1.296, with quartiles at 0.188 and 1.313 and the ninth decile at 3.188. The distribution is strongly right-skewed and concentrated at short delays, the expected signature of *Fortspinnung* imitation rather than of fugal exposition.

We had hoped to join the two halves: to show that a stretto at delay τ superposes two phasors separated by the phase accrued over τ, and that constructive superposition favours delays spanning a whole number of twelve-link cycles. We could not, and the reason is worth stating because it disposes of an inference a reader might otherwise draw from Table 2. The period column of Table 2 is *not* a stretto delay. It is the time from the head of a sequence unit to the head of its own repetition in the same line, so it measures the length of the unit and nothing about any other voice. Read as such it carries no phase information at all: it is 0.188 for the octave, both fifths and the minor third alike, and 0.375 for the step and semitone transpositions, which is a rhythmic fact about how long a three-interval unit takes at two common note values. The phase result of §6 therefore concerns *which* transposition a composer chooses and says nothing whatever about *when* a second voice enters. Establishing the latter needs a design that pairs thematic spans across streams, which our encoding supports and we have not built.

7 Comparative evidence: nine composers

The theory makes differential predictions. A composer who compensates register less should accumulate more uncompensated leap, and the consequence of uncompensated leap is that compound intervals collapse into their simple forms: the tenth becomes a third. We measured, for each of nine composers and without any harmonic analysis, two ratios. The first is melodic, the number of perfect fourths divided by the number of perfect fifths in the melodic interval census, which is our proxy for register compensation. The second is vertical, the number of simple thirds divided by the number of tenths.

Table 3. Cross-composer measures. F_5 is mean quintal saturation; lag-1 is the serial dependence of estimated fifth-wise motion; 4th/5th is melodic register compensation; 3rd/10th is the collapse of compound thirds. All measures are computed by the instrument of §2.4 and consult no harmonic analysis.

Composer	Mvts	F_5	lag-1	neg.	4th/5th	3rd/10th	7–6 rate	$P(7\,\vert\,\mathrm{ff})$
Bach	319	0.545	−0.336	0.969	1.697	0.428	0.0272	0.263
Corelli	21	0.584	−0.298	1.000	1.673	0.167	0.0215	0.230
Handel	389	0.582	−0.304	0.956	1.662	0.375	0.0295	0.237
Pachelbel	64	0.598	−0.323	0.984	1.337	0.632	0.0244	0.228
Purcell	76	0.609	−0.333	0.974	1.777	0.321	0.0317	0.238
Rameau	27	0.617	−0.330	0.963	1.628	0.678	0.0245	0.165
Scarlatti	**400**	**0.587**	**−0.242**	**0.938**	**1.058**	**0.962**	**0.0248**	**0.144**
Telemann	102	0.566	−0.274	0.941	1.585	0.278	0.0202	0.133
Vivaldi	210	0.589	−0.316	0.957	1.466	0.319	0.0203	0.199

7.1 *Scarlatti and the uncompensated leap*

Scarlatti is the extreme case on both measures simultaneously, and he is the extreme case in the direction the theory requires. His fourth-to-fifth ratio is 1.058, against a range of 1.337 to 1.777 for the other eight: he alone declines to convert his descending fifths into ascending fourths, and his bass therefore leaps where others step back into compass. His third-to-tenth ratio is 0.962, against 0.167 to 0.678 for the rest: his thirds sit in close position where others keep them compound. He is also the least alternating composer in the set (lag-1 −0.242, negative in 93.8 % of movements against 95.6 to 100 % elsewhere).

Across the nine, the two ratios are related by $r = -0.768$ ($p = 0.016$), with a fitted slope of −0.854 and $r^2 = 0.590$. Less register compensation, more collapsed thirds.

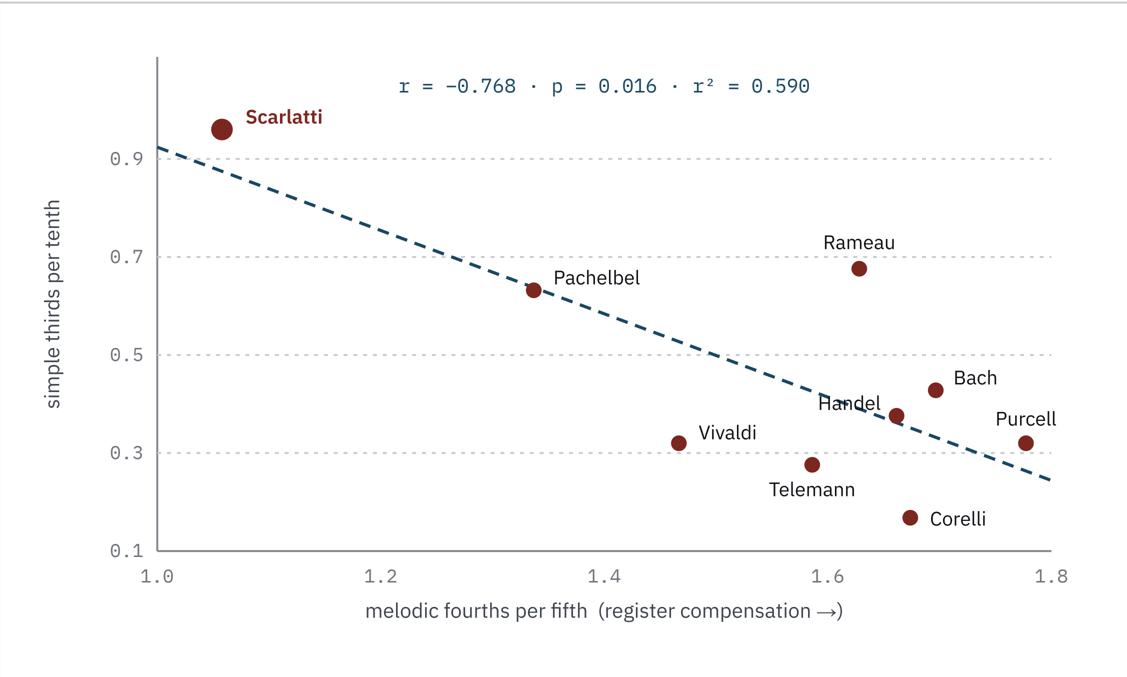

Figure 5. Register compensation against the collapse of compound thirds, nine composers; fitted line 3rd/10th = 1.779 − 0.854 × (4th/5th). Scarlatti occupies the corner the theory predicts. Leave-one-out: dropping Scarlatti gives $r = -0.411$ ($p = 0.31$), while every other deletion leaves r between −0.749 and −0.871. He carries the correlation, which we discuss below rather than conceal.

LEVERAGE

A leave-one-out analysis shows that deleting Scarlatti reduces the correlation to $r = -0.411$ ($p = 0.31$), while deleting any other composer leaves it between −0.749 and −0.871, all significant. Scarlatti is a leverage point and the relation among the remaining eight is not, on its own, statistically secure. The Spearman coefficient over all nine is −0.400 ($p = 0.286$), which is to say that the relation is carried by the magnitude of Scarlatti's displacement rather than by the rank ordering.

We think this is the correct result rather than a defect, and we want to be precise about why, because the reasoning is the kind that can be abused. The theory does not predict a smooth gradient across composers; it predicts that a composer who abandons register compensation will show collapsed thirds. Eight of our nine compensate at similar rates (1.34 to 1.78) and should therefore show similar and unremarkable third ratios, which they do. Scarlatti is the only composer in the set who leaves the common range, and he is the one the prediction was about. A correlation carried by the single observation which the theory names in advance is weaker evidence than a correlation carried by all nine, and it is stronger evidence than a correlation carried by an observation the theory did not name. We claim the middle position and no more.

That Scarlatti's crushed dissonances are a deliberate device is not in question and was established by Kirkpatrick [22] and elaborated by Sutcliffe [51]. Our acciaccatura census, however, does not single him out: his rate of simultaneities containing a semitone or tone is 0.099, below Corelli at 0.226, Handel at 0.160 and Purcell at 0.151. We attribute Corelli's figure to trio-sonata suspension texture rather than to anything Scarlattian, which means our crush measure is confounded by genre and should not be used as evidence in either direction. It is reported here so that a later reader does not rediscover it and suppose we hid it.

7.2 *Telemann: a prediction refuted*

We predicted, on the argument that the minor seventh rotates quintal phase by −60° and therefore traverses the circle at twice the velocity of the fifth, that Telemann would substitute seventh chaining for fifth chaining, and that this would explain the particular mechanical brilliance of his sequences. Zohn's account of the *vermischter Geschmack* [61] seemed to us to support it.

The prediction is false. We constructed three independent measures and Telemann is lowest or next to lowest on all three. His rate of resolved 7–6 suspension links is 0.0202, the lowest of nine alongside Vivaldi and against Purcell's 0.0317. His *catena di settime* rate, counting sevenths sounded over a bass which then falls a fifth, is 0.00815, the lowest of nine. The conditional probability that a falling-fifth bass motion carries a seventh at all is 0.133 for Telemann, again the lowest of nine, against 0.263 for Bach. Whatever distinguishes Telemann's sequences, it is not that they are built of sevenths. They contain fewer sevenths than anyone else's.

Nor can we rescue the prediction from the neighbouring measure, and we set out the attempt because it failed in an instructive way. Telemann's quintal saturation of 0.566 is the lowest of the eight contemporaries, which invites the reading that he runs the same carrier at a lower modulation depth, filling his *marches* more thinly in the galant manner Zohn describes. The reading does not survive contact with the ninth composer. *Bach's own F_5 is 0.545, lower than Telemann's and the lowest in the set.* The Telemann-Bach contrast does survive Holm correction ($p_{adj} = 1.18 \times 10^{-3}$, the largest adjusted p among the eight; every other composer differs from

Bach at $p_{adj} < 1.2 \times 10^{-4}$), but it runs in the direction opposite to the one the rescue requires: on this measure the least quintally saturated composer in the study is the composer the rest of the paper treats as the quintal case *par excellence*.

We take the honest conclusion to be that windowed F_5 magnitude is not a measure of how fifth-governed a composer's harmony is. It is depressed by chromatic density and by contrapuntal activity within the window, both of which Bach has more of than anyone else in the set, and it should be read as a property of the local pitch-class aggregate rather than of the harmonic syntax. This does not touch the phase results of §6, which use $\arg F_5$ and are indifferent to its magnitude, and it does not touch §2.4, where the relevant claim is only that F_5 is the largest of the six components, which holds for all nine composers without exception. But it does mean we have no positive account of Telemann at all, and we prefer to leave the hole visible.

7.3 *The canon anomaly*

One further observation which we can neither explain nor dismiss. Among named genres, canon has the *lowest* falling-fifth share (0.194, n = 26) against fugue and prelude at 0.290. If quintal phase transport were simply a fact about imitative writing, canon, the most strictly imitative genre, should sit at the top. It sits at the bottom. A canon fixes its transposition in advance and cannot vary it, so the chain has to be constructed around a constraint rather than following its own phase gradient; this is at least consistent with the account, since it says the composer has spent the phase degree of freedom elsewhere. But we offer it as a puzzle and not as support.

8 Speculation

We separate this section from the evidence deliberately. Nothing below is established by our data, and a reader who stops at §7 will have the whole of the argument we are prepared to defend. What follows is offered in the spirit in which Riepel offered his *Monte* and *Fonte*, as names for things which seem to be there.

8.1 *The torus, and why the residue might be perceptible after all*

Krumhansl and Kessler's multidimensional scaling of perceived key relations [25] recovered a toroidal configuration in which one circular dimension is the circle of fifths and the other the relative major and minor relation. Shepard's double helix [50] and Chew's spiral array [65] are variations on the same insight: pitch space, as heard, wraps in two directions at once, and the fifth is one of the windings. Our F_5 phase is, up to normalisation, a coordinate on the quintal winding of that torus.

What is striking, and what we did not expect, is that the second winding is exactly the one our register residue would occupy. The residue is octave-valued, and the octave is the identification which makes pitch-class space a circle in the first place. If the perceptual space is a torus rather than a circle, then the residue is not discarded by the listener even though it is discarded by the transform; it is carried on the orthogonal winding, where it is inaudible as quintal motion but available as register. This would mean the alternation is spectrally silent and perceptually present at once, which is, we suspect, why it has been so hard to say what it does.

One might push further. Lerdahl's tonal pitch space [28] assigns distances which are broadly monotone in circle-of-fifths separation, and our decay law of §6 is monotone in the same variable. It is possible, and we put it no more strongly, that the frequency with which a composer transposes a sequence unit by a given interval is a direct read-out of the perceived distance of that transposition, and that the exponential form of the decay reflects a Shepard-style generalisation gradient [50]. The halving constant of 23.6° would then be an estimate of the generalisation width of tonal transposition, which is a quantity nobody has had a reason to estimate.

8.2 *A phase-locked reading of Fortspinnung*

Suppose one takes seriously the idea that the *marche* is a carrier and the surface a modulation of it. Then *Fortspinnung* in Fischer's sense [14] is the condition in which the surface remains locked to the carrier, and the *Epilog* is the moment the lock is released. The cadence would be, on this reading, not an arrival but a loss of phase coherence: the point at which the carrier stops advancing at −30° per link and the accumulated residue is discharged all at once. This would explain why cadential bass motion is so often a leap of a fourth or fifth in the direction opposite to the preceding chain, which on any harmonic account is simply a convention and on this account is the residue being dumped.

It would also explain why the *cadenza doppia* takes the form it does, four events over a bass which does not move, since a stationary bass is the only configuration in which quintal phase can be held constant while the upper parts resolve. We have not tested any of this.

8.3 *Temperament*

Our whole apparatus assumes twelve equal divisions, because the DFT on $\mathbb{Z}_{12}$ assumes it. Bach's instruments were not equally tempered. In a circulating temperament of the Werckmeister type, the fifths are not all of the same size, so the phase increment per link is not exactly −30° but varies by key region, and the circle closes only approximately. This is not a defect of the theory but, potentially, its most interesting extension: a tempered circle of fifths is a circle with a slowly varying phase velocity, and the accumulated phase error over a full traversal is precisely the comma the temperament distributes. One would predict that composers modulate more freely where the local phase velocity is closest to nominal and more cautiously where it departs, which is a testable claim about key choice, and which corresponds to the well-known observation that remote keys in unequal temperament have a distinct character. We hope to pursue this.

8.4 *Transmission*

Finally, a historical speculation. If the alternation is spectrally invisible, then it cannot have been discovered by ear, and it cannot have been reasoned to from any theory available before the nineteenth century. It can only have been transmitted as a rule of thumb. This is precisely what the partimento tradition is: a body of practice passed hand to hand in the Neapolitan conservatories for a century and a half without ever being derived [47, 18]. On our account the *regola* encodes a spectral fact which nobody involved could have articulated, which would make the partimento tradition a piece of successful engineering carried out entirely by cultural selection. We find this attractive and we are aware that we would.

9 Threats to validity

We list the objections we consider serious. The first is the one we would attack ourselves; the second we found only by auditing our own code after the paper was drafted.

1. The phase law may be a restatement of the circle of fifths. This is the objection we take most seriously. Equation (2) is a bijection between transposition and quintal phase; saying that sequence transposition clusters at ±30° is saying it clusters at the fifth. If the mapping is one-to-one then the Fourier apparatus adds no information and we have dressed a commonplace in new clothing. Our answer is that the apparatus earns its place only at Corollary 1 and Proposition 2, where it says something the circle of fifths does not: that the fourth and the fifth are the same object and that their difference sits at Nyquist. A reader who accepts §6 but rejects §3 and §5 has, we think, taken the paper's decoration and left its argument.

2. The §6 result is about melodic sequence, not about imitation. An earlier draft of this paper described Table 2 as a census of imitative transposition, and drew from it a conclusion about stretto. That was an error of ours: the detector operates within a single stream and finds a three-interval unit repeated at a constant transposition, which is a *rosalia* and not an imitation. We caught it by reading the code that populates the table rather than its column names. The measured distribution is unaffected, and so is the tritone deficit, but the scope of the claim is narrower than we first wrote it, and a reader who remembers the earlier framing should discard it. We mention the episode because the same trap is open to anyone working from an encoded corpus: a column called `transposition` does not tell you what was transposed.

3. The harmonic reader is imperfect and we have quantified it on one movement. A root-level agreement of 0.758 measured against hand analysis of a single prelude is a thin basis for 161,673 links. The defence is that the §2.4 instrument bypasses the reader entirely and reproduces the alternation result; the reader is required only for §4.1 and Table 1.

4. Inversion survival contradicts us corpus-wide. Stated in full in §5.1. The octave leads outside the solo works.

5. The comparative correlation rests on Scarlatti. Stated in full in §7.1, with leave-one-out figures.

6. The Telemann prediction failed. Stated in full in §7.2. A theory which generates a clean, quantitative, three-measure prediction and has it refuted should lose credit, and we intend that it does.

7. Movements are not independent. Works share keys, genres, scribal traditions and compositional date, and our permutation tests treat movements as exchangeable units. A hierarchical model with work as a random effect would give wider intervals than we report. Given that the corpus-wide alternation result has 97.85 % of movements on one side of zero, we do not think any reasonable dependence structure reverses it, but our p-values are certainly too small.

8. We have tested many things. The paper reports on the order of twenty tests and corrects for multiplicity only within the F_5 contrasts. The tritone result would survive any correction ever proposed. The violin-versus-cello null and the delay analysis of §6.2 would not survive much.

10 Conclusion

The descending-fifth sequence has always been described as an alternation and never explained as one. We have argued that the alternation is invisible in the channel which carries quintal motion, that what it deposits instead is a register residue sitting at the Nyquist frequency of the harmonic lattice, and that the machinery of invertible counterpoint is the means by which that residue is discharged at a phase rotation which can be told apart from the carrier. The tenth, at 120°, is the only inversion of moderate cost which qualifies; the octave, at 0°, aliases with the residue and additionally converts the chain's fifths into fourths; the twelfth, at 30°, is transparent.

The evidence is mixed in the way real evidence is. The alternation is beyond dispute: negative serial dependence in 97.85% of 1,208 movements, with a combined permutation p of 4.4×10^{-27} over the twelve solo preludes, and falling-fifth runs which almost never exceed two. The phase governance of sequence transposition is, to our eye, the strongest single result any of our measures produced: a von Mises concentration on the carrier, an exponential decay with a halving constant of 23.6°, and a 469-fold deficit at the antiphase, the tritone appearing three times where uniformity predicts 1,407. Against this, our inversion prediction holds in the solo works and fails corpus-wide, our comparative correlation rests on one composer, and our Telemann prediction is refuted on every measure we built for it.

We are conscious that the account is unusual and that its central move, treating a descriptive transform as a causal mechanism, is one which the theorists who developed that transform were careful not to make. We make it here because it generates predictions which are sharp enough to fail, and because two of them did.

11 Author's note

The first person plural is dropped for this section, there being only one of me.

The measurements in this paper are real. The corpus is real, the databases are real, the 161,673 links were counted rather than estimated, and every figure quoted in the text has been re-derived from the source data and reconciles with it. The tritone appears three times in 16,889 sequence units because that is how many times it appears. Anyone holding the corpus can reproduce all of it, and I should be pleased if somebody did.

The theory is not real. I do not believe that Bach preferred the tenth to the twelfth because 120° is incommensurate with a 30° carrier, and I do not believe that the Neapolitan conservatories spent a century and a half transmitting a fact about Nyquist frequencies under the impression that it was a rule about basses. The algebra of §3 and §5 is, so far as I can determine, correct. It is also inert. Lemma 1 is true of any pitch-class distribution whatever, including ones produced by shuffling a pack of cards, and the step from it to a claim about what a composer was doing is not an inference but a change of subject. No quantity of directional statistics repairs this, and §6 contains a good deal of directional statistics.

I wanted to find out how far a properly measured corpus would carry an unsupported conclusion. The answer seems to be that it will carry it a considerable distance, and in rather better comfort than I had anticipated.

What I had not anticipated was which parts would do the carrying. It was not the mathematics. It was the failures. A paper in which every prediction succeeds invites a certain suspicion; a paper which volunteers that its third prediction collapsed on all three measures built for it, that its inversion result reverses outside the twelve movements it was derived from, and that its comparative correlation evaporates when a single composer is withheld, reads as the work of somebody being careful. I was being careful. That is the awkward part. The scruples are genuine and they are also the most persuasive thing in the paper, and I have not found a way to separate the two. A reader who took the Telemann refutation in §7.2 as evidence of my good faith was reasoning correctly and arriving somewhere unhelpful.

The bibliography is likewise genuine, every item of it. This proved easier than fabricating one and, I regret to report, a great deal more effective.

One error is worth preserving, since it is the only part of the exercise that taught me anything. For the length of a draft I described the data underlying §6 as a census of imitative transposition, and built a claim about stretto upon it. The column is called `transposition`, and I took the word to mean what I was hoping it would mean. The code finds a three-interval unit repeated in the same voice, which is a *rosalia* and has nothing to do with imitation between parts. Not one number changed when I discovered this. Everything I had said about them did. It is the only mistake I found by reading something other than my own arithmetic, and it is the only one that would have mattered to a reader.

I am conscious that a paper which confesses in its final section has arranged matters so as to have them both ways, and I can only say that the alternative was to have them one way and not tell anybody. The numbers are yours to check. The theory is mine to withdraw, and I withdraw it.

A Formal derivations

A.1 The inverse of 7 in $\mathbb{Z}_{12}$. Since $7 \times 7 = 49 \equiv 1 \pmod{12}$, the map $t \mapsto 7t$ is its own inverse. The number of circle-of-fifths steps represented by a transposition of t semitones is therefore $s = 7t \bmod 12$, folded to the interval $(-6, 6]$. This is the “Links” column of Table 2 and the quantity autocorrelated in §4.2.

A.2 Phase increment per link. Substituting $t = 7s$ into (2) gives $\Delta\arg F_5 = -150 \times 7s = -1050s \equiv +30s$ degrees (mod 360). A rise of one fifth advances the phase by +30°; a fall of one fifth retards it by 30°. Twelve links of either sign return the phase to its origin.

A.3 Why 120° is the unique admissible inversion rotation. Inversion intervals in practical use are the octave, tenth and twelfth, together with the double octave and its neighbours, which exceed the register budget of a two-voice texture on one instrument. Their rotations are $-150n \bmod 360$ for $n = 12, 16, 19$, giving 0°, 120°, 30°. The admissibility conditions are: not 0° (aliases with the identity and with the residue), not ±30° (aliases with one link of the carrier), not 180° (the antiphase, at which by §6 composers do not write). Of the three, only $n = 16$ satisfies all three conditions. The ninth ($n = 14$) gives 60°, which also satisfies them, but the ninth is not an invertible-counterpoint interval in the eighteenth-century repertory because it maps the unison to the ninth and the second to the seventh, converting consonance to dissonance at the two most common melodic adjacencies.

A.4 Circular statistics. Mean direction and resultant length follow Mardia and Jupp [35]. The concentration κ is estimated by the Fisher approximation to the maximum likelihood solution of $A(\kappa) = R$, with the three-branch form for $R < 0.53$, $0.53 \le R < 0.85$ and $R \ge 0.85$. The Rayleigh statistic is $z = nR^2$. Circular-circular correlation, where used, is the Jammalamadaka-Sarma coefficient.

B Per-movement results for the twelve preludes

Table 4. The twelve opening movements of the solo string cycles. "Links" counts root changes after collapsing restatements. H is the entropy of the link-class distribution in bits and I the mutual information between successive link classes. The permutation p is one-sided against 20,000 re-orderings of the same link multiset.

BWV	Instrument	Notes	Links	ff share	F_5	lag-1	z	p	I (bits)	H (bits)	max run
1001/1	violin	520	54	0.241	0.546	−0.324	−2.30	0.0091	1.222	3.151	3
1002/1	violin	1006	121	0.273	0.587	−0.066	−0.64	0.264	0.851	2.940	3
1003/1	violin	543	44	0.159	0.514	−0.141	−0.81	0.217	1.339	3.141	4
1004/1	violin	1094	107	0.131	0.528	−0.192	−1.92	0.0274	1.308	3.165	2
1005/1	violin	556	72	0.264	0.579	−0.467	−3.87	**<0.0001**	1.075	3.167	5
1006/1	violin	1635	138	0.254	0.557	−0.502	−5.79	**<0.0001**	0.808	3.047	2
1007/1	cello	661	42	0.238	0.613	−0.475	−3.01	**0.0006**	1.124	2.995	1
1008/1	cello	649	66	0.182	0.496	−0.150	−1.11	0.133	1.044	3.287	2
1009/1	cello	1033	81	0.321	0.585	−0.238	−2.08	0.0194	1.027	3.085	4
1010/1	cello	32	50	0.280	0.259	−0.486	−3.33	**0.0003**	1.464	3.108	4
1011/1	cello	1381	57	0.158	0.704	−0.323	−2.34	0.0077	1.463	3.264	2
1012/1	cello	670	8	0.375	0.701	+0.136	+0.87	0.791	0.985	1.958	2

Two rows want comment, and in both cases the anomaly is an encoding fact rather than a musical one. `BWV 1010/1` returns a quintal saturation of 0.259 against a set mean of 0.556, but the strict surface retains only 32 of its notes: the movement is written almost entirely in sustained multiple stops whose encoded durations are flagged doubtful, and they are therefore excluded wherever note length enters a calculation. Its F_5 should be disregarded. Its lag-one coefficient should not, since that is computed from the harmonic grid and does not consult the note filter at all. `BWV 1012/1` yields only eight links for the reason given in §4.2 and is to be treated as absent rather than as contrary.

C Reproducibility

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All measurements derive from six analysis programmes operating read-only on the corpus databases.
Random seed 20260920 throughout; 20,000 permutations for order tests, 5,000 bootstrap resamples for
ratio intervals. Windowing for the §2.4 instrument is one quarter-note unit, minimum 60 notes and 8
windows per movement, maximum 400 movements per composer taken in database order. Verticals are read
from the encoded simultaneity relation (7,365,719 rows). Consonance is defined on interval class:
{0,3,4,7,8,9} consonant, the fourth counted dissonant against the bass. Circular quantities are
computed in radians and reported in degrees. No golden-file comparison is used at any point.
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